

# On non-linear CMB temperature anisotropy from gravitational perturbations

Xian Gao\*

*Kavli Institute for Theoretical Physics China,  
Key Laboratory of Frontiers in Theoretical Physics,  
Institute of Theoretical Physics, Chinese Academy of Sciences,  
No.55, ZhongGuanCun East Road,  
HaiDian District, Beijing, 100190, China*

Non-linear CMB temperature anisotropies up to the third-order on large scales are calculated. On large scales and in the Sachs-Wolfe limit, we give the explicit expression for the observed temperature anisotropy in terms of the primordial curvature perturbation up to third-order. We derived the final bispectrum and trispectrum of anisotropies and the corresponding non-linear parameters, in which the contributions to the observed non-Gaussianity from primordial perturbations and from the non-linear mapping from primordial curvature perturbation to the temperature anisotropy are transparently separated.

PACS numbers: Valid PACS appear here

## I. INTRODUCTION

In the past few years, extensive attention has been attracted to the investigation of cosmological perturbations [3] beyond the linear order. The importance of studying non-linear perturbations comes two aspects. Firstly, forthcoming experiments of Cosmic Microwave Background (CMB) and Large-scale Structure (LSS) will be able to detect the non-linear structures in these perturbations. The observational detection of the non-linearities through the statistical non-Gaussianity (see [4] for a comprehensive review and [2] for a review of recent observational progress) of perturbations has become one of the primary targets of the cosmology. On the other hand, non-linearities, which encode the *interactions* in the early universe, would definitely bring us new understandings of both the early universe and the fundamental physics.

A large amount of efforts have been devoted to the calculation of the statistics of curvature perturbation  $\zeta$ , like primordial bispectrum and trispectrum on large scales, pioneered by Maldacena [57]<sup>1</sup>. However, these are not the observed non-Gaussianities of (e.g.) CMB temperature anisotropies  $\frac{\Delta T}{T}$ . Conventionally, one may use the linear order relation  $\frac{\Delta T}{T} = -\frac{1}{3}\Phi = -\frac{1}{5}\zeta$  to evaluate the angular bispectrum or trispectrum of CMB, assuming the contributions from the second-order or secondary effects are negligible comparing to the primordial ones [1, 2, 6]. However, in light of increasingly precise observations, a full treatment of the higher-order radiation transfer functions of the CMB anisotropies is needed, which will allow us to make definite prediction of CMB non-Gaussianities (see [5] for a recent review for non-Gaussianity on the CMB).

The research on non-linear temperature anisotropy due to gravitational perturbations was pioneered by [8–10], in which the second-order generalization of Sachs-Wolfe (SW) effect and Integrated Sachs-Wolfe (ISW) effect were derived. These results were extended in [13, 14, 18–20] where the second-order radiation transfer function on large scales was calculated, and in [15] where the general expression for anisotropy due to gravitational perturbations up to the third-order and in [16] where an elegant and non-perturbative non-linear anisotropy were got. The non-linear anisotropies have also been analyzed in [11, 12] base on the covariant approach to cosmological perturbations (see [54] for a recent review). Various secondary contributions to the non-Gaussianities have also been extensively studied, including the weak gravitational lensing and its correlation with ISW effect [17, 21–26], which is expected to be the dominant contamination of  $f_{\text{NL}}^{\text{local}}$ , correlation between lensing and Sunyaev-Zel'dovich (SZ) effect [22], inhomogeneous recombination [27–29], small-scale dark matter clustering [30, 34]. A systematic treatment of the transfer function on all scales, which involves solving the full Boltzmann equations through the recombination phase and then from the surface of last-scattering until today, has been performed in [31–38] at second-order.

In this note, we calculate the CMB temperature anisotropy up to the third-order in primordial curvature perturbation  $\zeta$ , which can be viewed as non-linear generalization of linear-order relation  $\Delta T/T = -\zeta/5$ . We follow the same strategy in [8–10, 13, 15]. First, we calculate the gravitational redshift of a given photon from the emission surface to today's observer, which will give the observed anisotropy in terms of metric perturbations  $\frac{\Delta T}{T} = \frac{\Delta T}{T}[\Phi, \Psi, \dots]$ . Then by using the conservation of curvature perturbation  $\zeta$ , in the large-scale limit, we determine the initial conditions for the metric perturbations in matter-dominated era, more precisely, the values  $\Phi_e = \Phi_e[\zeta]$  and  $\Psi_e = \Psi_e[\zeta]$  on the emission surface. Combining these two procedures will give the final non-linear mapping from  $\zeta$  to  $\frac{\Delta T}{T}$ :  $\frac{\Delta T}{T} = \frac{\Delta T}{T}[\zeta]$ . Together with previous results of primordial non-Gaussianities of  $\zeta$  got in the literature, our formalism is ready to make prediction of the final observed CMB non-Gaussianity. Obviously, since we match the metric perturbations in matter era directly to those on the emission surface after last-scattering, our formalism includes only the gravitational redshift of photons, without considering the dynamics of photon-baryon plasma. Thus our result is valid only for large-scale anisotropies, which enter the horizon after decoupling.

\*Electronic address: gaoxian@itp.ac.cn

<sup>1</sup> See e.g. [58] for a recent short review of methods and techniques in calculating and analyzing primordial non-Gaussianities from inflationary models and references therein.

This note is organized as follows. In the following section, we describe the temperature anisotropy induced by the gravitational perturbations from the emission surface to the observers. We give the general expression for the temperature anisotropy up to the third-order in metric perturbations. In the next section, by using the conserved curvature perturbation and taking the large-scale limit, we determine the initial conditions for metric perturbations in matter-dominated era. Then we give the non-linear mapping from primordial curvature perturbation to the temperature anisotropy, in the Sachs-Wolfe limit. Finally we give a short conclusion.

## II. FORMALISM

After decoupling, the CMB photon density remains as Planck distribution, which is determined by a single parameter — the photon temperature. The temperature shift of a Planck distribution of photons is exactly proportional to the energy shift of any given photon, i.e.  $T/\omega = \text{const}$ , if there is no collision, which implies

$$\frac{T_f}{T_i} = \frac{\omega_f}{\omega_i}, \quad (2.1)$$

where  $T_f$  and  $T_i$  are the final and initial temperature of the Planck distribution respectively,  $\omega_f$  and  $\omega_i$  are the final and initial energy of a given photon respectively. (2.1) is exact, which implies that in order to get the change in the temperature, we need to evaluate the change in the energy of a (any) given photon, which in our question is nothing but its gravitational redshift. Thus, the question becomes to follow the geodesic equation of a given photon from the last-scattering surface to us, taking the inhomogeneous spacetime background into account.

We work in the generalized Poisson gauge

$$ds^2 = a^2 g_{\mu\nu} dx^\mu dx^\nu = a^2 (-e^{2\Phi} d\eta^2 + 2\sigma_i d\eta dx^i + e^{-2\Psi} e^{\gamma_{ij}} dx^i dx^j), \quad (2.2)$$

where<sup>2</sup>  $\sigma_{i,i} = 0$ ,  $\gamma_{ij}$  is symmetric and satisfies  $\gamma_{ij,i} = \gamma_{ii} = 0$  (thus  $\det e^{\gamma_{ij}} = 1$ ),  $a$  is the scale factor. The energy of a given photon with physical momentum  $P^\mu$  measured by an observer with 4-velocity  $u^\mu \equiv v^\mu/a$  (normalized as  $a^2 g_{\mu\nu} u^\mu u^\nu = g_{\mu\nu} v^\mu v^\nu = -1$ ) is<sup>3</sup>

$$\omega = -a^2 g_{\mu\nu} u^\mu P^\nu = -g_{\mu\nu} u^\mu p^\nu, \quad (2.3)$$

where  $p^\mu$  is the momentum associated with the conformal metric  $g_{\mu\nu}$  (note  $p^\mu = P^\mu/a^2$ ). Under the perturbed metric (2.2) and using normalization for  $u^\mu$  and  $p^\mu$  to express  $u^0$  and  $p^0$  in terms of  $u^i$  and  $p^i$ ,  $\omega$  in terms of all relevant components takes the form:

$$\omega = \frac{1}{a} \left( \sqrt{\tilde{g}_{ij} p^i p^j} \sqrt{\tilde{g}_{ij} v^i v^j} + 1 - \tilde{g}_{ij} v^i p^j \right), \quad (2.4)$$

with

$$\tilde{g}_{ij} \equiv e^{-2\Psi} e^{\gamma_{ij}} + e^{-2\Phi} \sigma_i \sigma_j. \quad (2.5)$$

From (2.4), on the background level  $\bar{\omega} = |\bar{p}^i|/a = \bar{p}^0/a$ , which we will normalize by setting  $\bar{p}^0 = 1$  in the following. The explicit expression for  $u^i$  (or  $v^i$ ) requires details of the dynamics during recombination.

Assuming the “intrinsic” photon temperature anisotropy at emission point  $x_e$  in the direction  $\mathbf{n}_e$  takes the form  $T(\eta_e, x_e^i, n_e^i) = \bar{T}(\eta_e, \bar{x}_e^i, n^i) e^{\tau(\eta_e, x_e^i, n_e^i)}$ , where  $\eta_e$  is the constant conformal time of emission, e.g. when the last-scattering takes place. The temperature measured by an observer at  $x_o$  and in the direction  $\mathbf{n}$  is given by

$$\frac{\Delta T}{T}(x_o^i, n^i) \equiv \frac{T(x_o^i, n^i) - \bar{T}(x_o^i, n^i)}{\bar{T}(x_o^i, n^i)} = \frac{a_o \omega_o(\eta_o, x_o^i, p_o^\mu)}{a_e \omega_e(\eta_e, x_e^i, p_e^\mu)} e^{\tau(\eta_e, x_e^i, n_e^i)} - 1, \quad (2.6)$$

where we have used  $\bar{T}_o = \frac{\bar{\omega}_o}{\bar{\omega}_e} \bar{T}_e = \frac{a_e}{a_o} \bar{T}_e$ . From (2.6) it is clear that the observed anisotropy comes from two aspects: the intrinsic anisotropy  $\tau$  on the emission surface which depends on the dynamics during recombination and the gravity theory, the other is the gravitational redshift-induced anisotropy from the emission surface to the observer  $a_o \omega_o / (a_e \omega_e)$ , which is purely kinetic and is independent of the theory of gravitation. The intrinsic anisotropy  $\tau$  is highly model-dependent and a full treatment needs solving the set of Boltzmann equations up to the third-order which is beyond the scope of this note. On super-Horizon (the sound horizon on the last-scattering surface) scales where microscopic physics is irrelevant, a simple and non-perturbative expression has been got [16]:  $\tau = -\frac{2}{3}\Phi$ , which is adequate for our purpose.

In the following, first we derive the gravitational redshift-induced anisotropy up to the third-order in terms of metric perturbations in (2.2), then use the Einstein equation to determine the initial condition at matter-dominated era, i.e. the values of metric perturbations in terms of the conserved curvature perturbation  $\zeta$  on large scales.

<sup>2</sup> Here and in what follows, two repeated lower or upper spatial indices are contracted by  $\delta_{ij}$

<sup>3</sup> As is well-known, metric  $\tilde{g}_{\mu\nu} \equiv e^{2\alpha} g_{\mu\nu}$  and  $g_{\mu\nu}$  have the same null geodesics, but parameterized by different affine parameters  $\tilde{\lambda}$  and  $\lambda$  with relation  $d\lambda = e^{2\alpha} d\tilde{\lambda}$ . Thus the energy in metric  $\tilde{g}_{\mu\nu}$  can be expressed as  $\omega \equiv -\tilde{g}_{\mu\nu} u^\mu \tilde{p}^\nu = -g_{\mu\nu} u^\mu p^\nu$ , where  $\tilde{p}^\mu = dx^\mu/d\tilde{\lambda}$  and  $p^\mu = dx^\mu/d\lambda$  are momentum in metric  $\tilde{g}_{\mu\nu}$  and  $g_{\mu\nu}$  respectively.

### A. Perturbed photon energy

After decoupling, the photons propagate freely and thus the physical content of its Boltzmann equation is completely encoded in the photon geodesic equation, which is much simpler to deal with. Null geodesics in perturbed spacetime has been investigated long before [7–10, 13, 15]. Although the whole thing we need to do is simply to follow the redshift of a photon, in a practical calculation, the complexities arise from several aspects. First, the energy  $\omega$  should be evaluated at the “real” emission point  $x^i(\lambda_e)$  rather than at the “virtual image” at  $\bar{x}^i(\bar{\lambda}_e) \equiv (\lambda_e - \lambda_o)n^i$ . Here  $\lambda$  is the affine parameter along the photon geodesics,  $\lambda_e$  ( $\lambda_o$ ) is the corresponding values at emission surface (observer). Second, to determine the real position of emission we need to follow the photon geodesics, which we are able to solve only perturbatively.

The expansion of frequency  $\omega$  around the background emission point  $x_o^i$  and the direction  $n^i \equiv -\bar{p}^i$  is straightforward [8–10, 14, 15]. Here we simply report the corresponding expansions of (2.4) up to the third-order in the metric perturbations (2.2), which involve the spatial components  $v^i$  and  $p^i$ . At the linear-order:

$$a\omega_{(1)} = \left(v_{(1)}^i - p_{(1)}^i\right) n^i - \Psi + \frac{1}{2}\gamma_{ij}n^in^j, \quad (2.7)$$

where throughout this note we take the expansion of variable  $Q$  as  $Q = \bar{Q} + Q_{(1)} + Q_{(2)} + Q_{(3)} + \dots$ . At second-order:

$$\begin{aligned} a\omega_{(2)} = & \left(v_{(2)}^i - p_{(2)}^i + x_{(1)}^0 p_{(1)}^i\right) n^i + \frac{1}{2}\Psi^2 + \Psi n^i \left(p_{(1)}^i - 2v_{(1)}^i\right) + \frac{1}{2}\left(p_{(1)}^i - v_{(1)}^i\right)^2 - \frac{1}{2}\left(n^i p_{(1)}^i\right)^2 \\ & + \frac{1}{4}\left[\gamma_{ij}^2 - 2\left(\Psi - n^k p_{(1)}^k + \frac{1}{4}\gamma_{kl}n^k n^l\right)\gamma_{ij}\right] n^i n^j + \gamma_{ij}n^i \left(v_{(1)}^j - p_{(1)}^j\right) \\ & + \left(x_{(1)}^k + x_{(1)}^0 n^k\right) \partial_k \left(v_{(1)}^i n^i - \Psi + \frac{1}{2}\gamma_{ij}n^i n^j\right) + \frac{1}{2}\left(\sigma_i n^i\right)^2. \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} a\omega_{(3)} = & n^i \left[v_{(3)}^i - p_{(3)}^i + \left(x_{(1)}^j + x_{(1)}^0 n^j\right) \left(\partial_j v_{(2)}^i + \frac{1}{2}\left(x_{(1)}^k + x_{(1)}^0 n^k\right) \partial_j \partial_k v_{(1)}^i\right) + \left(x_{(2)}^j - x_{(1)}^0 p_{(1)}^j\right) \partial_j v_{(1)}^i\right. \\ & + x_{(1)}^0 p_{(2)}^i + \left(x_{(2)}^0 - x_{(1)}^0 p_{(1)}^0\right) \left(n^j \partial_j v_{(1)}^i + \dot{p}_{(1)}^i\right) - \frac{1}{2}\left(x_{(1)}^0\right)^2 \ddot{p}_{(1)}^i] \\ & + \left[\left(\Psi - n^j p_{(1)}^j + \frac{1}{2}\gamma_{kl}n^k n^l\right) n^i - \gamma_{ij}n^j + p_{(1)}^i - v_{(1)}^i\right] \left(p_{(2)}^i - x_{(1)}^0 p_{(1)}^i\right) \\ & + \left(v_{(1)}^i - p_{(1)}^i + \gamma_{ij}n^j - 2\Psi n^i\right) \left(v_{(2)}^i + \left(x_{(1)}^j + x_{(1)}^0 n^j\right) \partial_j v_{(1)}^i\right) \\ & - \left[\left(x_{(2)}^k + x_{(2)}^0 n^k - x_{(1)}^0 p_{(1)}^0 n^k - x_{(1)}^0 p_{(1)}^k\right) \partial_k + \frac{1}{2}\left(x_{(1)}^k + x_{(1)}^0 n^k\right) \left(x_{(1)}^l + x_{(1)}^0 n^l\right) \partial_k \partial_l\right] \left(\Psi - \frac{1}{2}\gamma_{ij}n^i n^j\right) \\ & + \frac{1}{2}n^k p_{(1)}^k \left(p_{(1)}^i\right)^2 - \frac{1}{2}\left(n^i p_{(1)}^i\right)^3 - \Phi \left(\sigma_i n^i\right)^2 - \frac{1}{6}\Psi^3 - \Psi^2 \left(\frac{1}{2}n^i p_{(1)}^i - \frac{1}{4}\gamma_{ij}n^i n^j - 2v_{(1)}^i n^i\right) \\ & + \Psi \left[\frac{1}{2}\left(n^i p_{(1)}^i\right)^2 - \frac{3}{2}\left(v_{(1)}^i\right)^2 + \frac{1}{2}\left(4v_{(1)}^i - p_{(1)}^i\right) p_{(1)}^i\right. \\ & + \frac{1}{8}\left[\left(\gamma_{kl}n^k n^l - 4n^k p_{(1)}^k\right) \gamma_{ij} - 2\gamma_{ij}^2\right] n^i n^j + \gamma_{ij}n^i \left(p_{(1)}^j - 2v_{(1)}^j\right) + \frac{1}{2}\left(\sigma_i n^i\right)^2] \\ & + \frac{1}{16}\left(\gamma_{ij}n^i n^j\right) \left[12\left(n^k p_{(1)}^k\right)^2 - 4\left(p_{(1)}^k\right)^2 + 8\gamma_{kl}n^k p_{(1)}^l + \left(\gamma_{kl}n^k n^l - 6n^k p_{(1)}^k\right) \left(\gamma_{mn}n^m n^n\right) - 2\gamma_{kl}^2 n^k n^l\right] \\ & + \frac{1}{2}\left[\left(p_{(1)}^i - 2n^k p_{(1)}^k n^i\right) p_{(1)}^j + v_{(1)}^i \left(v_{(1)}^j - 2p_{(1)}^j\right)\right] \gamma_{ij} + \frac{1}{4}\left(n^k p_{(1)}^k n^j + 2v_{(1)}^j\right) \gamma_{ij}^2 n^i + \frac{1}{12}\gamma_{ij}^3 n^i n^j \\ & + \left(x_{(1)}^k + x_{(1)}^0 n^k\right) \left\{\left[\Psi - \frac{1}{2}\gamma_{ij}n^i n^j + n^i \left(p_{(1)}^i - 2v_{(1)}^i\right)\right] \partial_k \Psi - \frac{1}{8}\partial_k \left(\gamma_{ij}n^i n^j\right)^2\right. \\ & + \left(n^i \left(v_{(1)}^j - p_{(1)}^j\right) - \frac{1}{2}\left(\Psi - n^l p_{(1)}^l\right) n^i n^j\right) \partial_k \gamma_{ij} + \left(\frac{1}{2}\gamma_{il} \partial_k \gamma_{lj} + \sigma_i \partial_k \sigma_j\right) n^i n^j\} \\ & + \sigma_i \sigma_j n^i \left(v_{(1)}^j - p_{(1)}^j\right) + \frac{1}{4}\left(2n^i p_{(1)}^i - \gamma_{ij}n^i n^j\right) \left(\left(\sigma_k n^k\right)^2 - \left(v_{(1)}^k\right)^2\right). \end{aligned} \quad (2.9)$$

In (2.7)-(2.9), a dot denotes derivative with respect to  $\lambda$  and subscripts “ $(i)$ ” denote the orders in metric perturbations and all quantities are evaluated on the background emission point (virtual image)<sup>4</sup>. In deriving the above results, we have used the

<sup>4</sup> (2.7)-(2.9) are essentially equal to (e.g.) eq.(2.17)-(2.19) in [15]. The differences come from 1) here we have replaced  $v^0$  and  $k^0$  in terms of  $v^i$ ,  $k^i$  and metric perturbations through the constraints  $v^2 = -1$  and  $k^2 = 0$ , 2) we use  $\Phi$  and  $\Psi$  rather than  $\phi$  and  $\psi$  which are defined as  $1 + 2\phi \equiv e^{2\Phi}$ ,  $1 - 2\psi \equiv e^{-2\Psi}$  and 3) at this point we have not expanded  $\Phi = \Phi_{(1)} + \Phi_{(2)} + \Phi_{(3)}$  etc.

expansion of  $\lambda_e$  around its background value:  $\lambda_e = \bar{\lambda}_e + \lambda_{(1)} + \lambda_{(2)}$  where  $\lambda_{(1)} = -x_{(1)}^0(\bar{\lambda}_e)$  and  $\lambda_{(2)} = -x_{(2)}^0(\bar{\lambda}_e) + x_{(1)}^0(\bar{\lambda}_e)p_{(1)}^0(\bar{\lambda}_e)$ . This can be got by perturbing  $x^0(\lambda_e) = \bar{x}^0(\bar{\lambda}_e) \equiv \eta_e$ , which is the definition of the emission surface as intersection of past light-cone of the observer and the spatial hypersurface at constant  $\eta_e$ .

## B. Photon geodesics

(2.7)-(2.9) can be fully determined when the perturbed photon geodesics is solved. Geodesic equation in the conformal metric  $g_{\mu\nu}$  is

$$\dot{p}^\mu + \Gamma_{\rho\sigma}^\mu p^\rho p^\sigma = 0, \quad (2.10)$$

where  $p^\mu = dx^\mu/d\lambda$ , a dot denotes  $d/d\lambda$ ,  $\Gamma_{\rho\sigma}^\mu$  is the connection associated with the conformal metric  $g_{\mu\nu}$ .

Since in (2.7)-(2.9) we have expressed the photon energy in terms of spatial momentum  $p^i$ , it is adequate to solve  $p^i$  perturbatively [7–10, 13, 15]. In order to make (2.10) a close set of equations for  $x^i$ , we also need the expression for  $x^0$  in terms of  $x^i$ . This can be done by perturbing the constraint  $p_\mu p^\mu = 0$  around the background geodesics, which yields (up to the second-order in metric perturbation)

$$p_{(1)}^0 = -n^i p_{(1)}^i - A, \quad (2.11)$$

with

$$A \equiv \Phi + \Psi + \sigma_i n^i - \frac{1}{2} \gamma_{ij} n^i n^j, \quad (2.12)$$

and

$$\begin{aligned} p_{(2)}^0 = & -n^i p_{(2)}^i + \frac{1}{2} (\delta_{ij} - n^i n^j) p_{(1)}^i p_{(1)}^j + \left[ \left( \Phi + \Psi + \frac{1}{2} \gamma_{jk} n^j n^k \right) n^i + \sigma_i - \gamma_{ij} n^j \right] p_{(1)}^i \\ & - x_{(1)}^\mu \partial_\mu A + \frac{1}{2} (\Phi + \Psi)^2 + 2\Phi \sigma_i n^i - \left( \frac{1}{2} (\Phi + \Psi) \gamma_{ij} - \frac{1}{2} \sigma_i \sigma_j - \frac{1}{4} \gamma_{ij}^2 \right) n^i n^j - \frac{1}{8} (\gamma_{ij} n^i n^j)^2. \end{aligned} \quad (2.13)$$

Integration of (2.11) and (2.13) with respect to  $\lambda$  along the background geodesics  $\bar{x}^\mu(\lambda)$  will give  $x_{(1)}^0$  and  $x_{(2)}^0$  in terms of  $x_{(1)}^i$  and  $x_{(2)}^i$  respectively.

Having deriving the general expressions for perturbed photon energy (2.7)-(2.9), in the following, we restrict ourselves to the large-scale limit. This is mainly because that in our formalism, to eventually determine the observed anisotropy in terms of conserved primordial curvature perturbation  $\zeta$ , we use the values of metric perturbations at matter-dominated era as initial conditions for gravitational redshift of photons rather than for the full Boltzmann equations. Thus, our results are only valid for the large-scale perturbation modes, which enter the horizon after decoupling and never affected by the microphysics. In the following we neglect the vector and tensor metric perturbations, not only because on large scales vector and tensor modes are subdominant but also calculation involving these modes up to the third-order is rather cumbersome. We also assume the observer is comoving with the emission point, i.e  $v^i = 0$ . Under these assumptions, a non-perturbative approach to the null geodesics has also been developed in [15].

Following the logic in [7–10, 13, 15], for spatially-flat FRW background, the set of perturbed geodesic equations are  $\dot{p}_{(n)}^i = \ddot{x}_{(n)}^i = f_{(n)}^i$ , with

$$f_{(1)}^i = -\Gamma_{\rho\sigma(1)}^i \bar{p}^\rho \bar{p}^\sigma, \quad (2.14)$$

$$f_{(2)}^i = -\Gamma_{\rho\sigma(2)}^i \bar{p}^\rho \bar{p}^\sigma - 2\Gamma_{\rho\sigma(1)}^i \bar{p}^\rho \dot{x}_{(1)}^\sigma - x_{(1)}^\lambda \partial_\lambda \Gamma_{\rho\sigma(1)}^i \bar{p}^\rho \bar{p}^\sigma, \quad (2.15)$$

$$\begin{aligned} f_{(3)}^i = & -\Gamma_{\rho\sigma(3)}^i \bar{p}^\rho \bar{p}^\sigma - 2\Gamma_{\rho\sigma(1)}^i \bar{p}^\rho \dot{x}_{(2)}^\sigma - 2\Gamma_{\rho\sigma(2)}^i \bar{p}^\rho \dot{x}_{(1)}^\sigma - \Gamma_{\rho\sigma(1)}^i \dot{x}_{(1)}^\rho \dot{x}_{(1)}^\sigma \\ & - 2x_{(1)}^\lambda \partial_\lambda \Gamma_{\rho\sigma(1)}^i \bar{p}^\rho \dot{x}_{(1)}^\sigma - \left( x_{(1)}^\lambda \partial_\lambda \Gamma_{\rho\sigma(2)}^i + x_{(2)}^\lambda \partial_\lambda \Gamma_{\rho\sigma(1)}^i + \frac{1}{2} x_{(1)}^\lambda x_{(1)}^\tau \partial_\lambda \partial_\tau \Gamma_{\rho\sigma(1)}^i \right) \bar{p}^\rho \bar{p}^\sigma. \end{aligned} \quad (2.16)$$

The perturbed Christoffel symbol can be read from (B7). After some manipulations, we can solve, at first-order in  $\Phi$  and  $\Psi$ ,

$$p_{(1)}^i = -2n^i \Psi - I_1^i, \quad (2.17)$$

$$p_{(1)}^0 = -2\Phi + I_1, \quad (2.18)$$

with

$$I_1^i = \int_{\lambda_o}^\lambda d\tilde{\lambda} \partial_i A, \quad I_1 = \int_{\lambda_o}^\lambda d\tilde{\lambda} A', \quad \text{where } A \equiv \Phi + \Psi. \quad (2.19)$$

Here and in the following we will frequently use the trick: (e.g.)  $\dot{\Psi} = (\Psi' - n^i \partial_i \Psi) + \dots$ . At second-order [8–10]:

$$p_{(2)}^i = -2n^i \left( x_{(1)}^\mu \partial_\mu \Psi + \Psi^2 \right) - 2\Psi I_1^i + I_2^i, \quad (2.20)$$

$$p_{(2)}^0 = 2\Phi^2 - 2x_{(1)}^\mu \partial_\mu \Phi - 2I_1 \Phi + I_1^2 + I_2, \quad (2.21)$$

with

$$I_2^i \equiv \int_{\lambda_o}^{\lambda} d\tilde{\lambda} \left[ 2(\Phi - I_1) - x_{(1)}^\mu \partial_\mu \right] \partial_i A, \quad (2.22)$$

$$I_2 \equiv \int_{\lambda_o}^{\lambda} d\tilde{\lambda} \left( x_{(1)}^\mu \partial_\mu A' - 2\Phi A' \right), \quad (2.23)$$

At the third-order [15],

$$\begin{aligned} p_{(3)}^i = & -\frac{4}{3}n^i \Psi^3 - 2n^i \left( x_{(2)}^\lambda \partial_\lambda \Psi + \frac{1}{2}x_{(1)}^\lambda x_{(1)}^\tau \partial_\lambda \partial_\tau \Psi + 2\Psi x_{(1)}^\mu \partial_\mu \Psi \right) \\ & - 2\Psi^2 I_1^i + 2\Psi I_2^i - 2I_1^i x_{(1)}^\lambda \partial_\lambda \Psi + I_3^i, \end{aligned} \quad (2.24)$$

with

$$\begin{aligned} I_3^i = & \int_{\lambda_o}^{\lambda} d\tilde{\lambda} \left[ 2(\Phi - \Psi - I_1) x_{(1)}^\mu \partial_\mu \partial_i A - x_{(2)}^\mu \partial_\mu \partial_i A - \frac{1}{2}x_{(1)}^\lambda x_{(1)}^\tau \partial_\lambda \partial_\tau \partial_i A \right. \\ & \left. - \left( 2(\Phi^2 - 2\Psi\Phi) + 3I_1^2 + 2I_2 + 4(\Psi - \Phi)I_1 - 2x_{(1)}^\mu \partial_\mu \Phi \right) \partial_i A \right]. \end{aligned} \quad (2.25)$$

Finally, after plugging (2.17)-(2.24) into (2.7)-(2.9), the perturbed photon frequency up to the third-order in  $\Phi$  and  $\Psi$  is given by

$$a\omega_{(1)} = -\Phi + I_1, \quad (2.26)$$

$$a\omega_{(2)} = \frac{1}{2}\Phi^2 + I_2 - \Phi I_1 + I_1^2 - x_{(1)}^0 A' - \left( x_{(1)}^i + x_{(1)}^0 n^i \right) \partial_i \Phi, \quad (2.27)$$

and

$$\begin{aligned} a\omega_{(3)} = & -\frac{\Phi^3}{6} + \frac{\Phi^2 I_1}{2} + I_1^3 + I_3 + I_1 (2I_2 - x_{(1)}^0 A') + \Phi \left( x_{(1)}^0 A' - I_1^2 - I_2 \right) - x_{(2)}^0 A' - \frac{1}{2} \left( x_{(1)}^0 \right)^2 A'' \\ & - \partial_i \Phi \left[ n^i \left( (A + \Psi - 2x_{(1)}^0) x_{(1)}^0 + x_{(2)}^0 \right) + x_{(2)}^i + x_{(1)}^0 I_1^i - x_{(1)}^i (\Phi - I_1) \right] \\ & - \frac{1}{2} \partial_i \partial_j \Phi \left( x_{(1)}^i + n^i x_{(1)}^0 \right) \left( x_{(1)}^j + n^j x_{(1)}^0 \right) - \frac{1}{2} x_{(1)}^0 \left( 4x_{(1)}^0 n^i \partial_i \Phi' + \left( n^i x_{(1)}^0 + 2x_{(1)}^i \right) \partial_i A' \right). \end{aligned} \quad (2.28)$$

In (2.26)-(2.28), all quantities are evaluated at the background emission point at  $\bar{x}^i(\bar{\lambda}_e) \equiv (\lambda_e - \lambda_o)n^i$ .

### III. TEMPERATURE ANISOTROPY UP TO THE THIRD-ORDER

#### A. Relation with primordial perturbations

Having derived the perturbed photon energy in terms of the metric perturbations at emission surface, the next goal is to relate the temperature anisotropy to the primordial curvature perturbation  $\zeta$ , which encodes the information in the very early universe and is the most frequently used variable in evaluating the primordial non-Gaussianities in the literature. It is well-known that on large scales and for adiabatic perturbation, there is a non-perturbative and conserved quantity which can be identified as non-linear curvature perturbation in uniform-density slices [45–48], defined as<sup>5</sup>

$$\zeta \equiv -\Psi + \frac{1}{3} \int_{\bar{\rho}}^{\rho} \frac{d\tilde{\rho}}{3(\tilde{\rho} + \bar{p})}. \quad (3.1)$$

Conserved and gauge-invariant  $\zeta$  beyond the linear theory has also been constructed perturbatively in [49–51]. The conservation of  $\zeta$  will allow us to relate the primordial era when modes exit the horizon during inflation and the era when modes re-enter the

<sup>5</sup> The existence of non-perturbative conserved perturbation has also been derived by using covariant formalism [52, 53], where the corresponding quantity is defined as a co-vector:  $\zeta_a = \partial_a \alpha - \frac{\dot{\alpha}}{\bar{\rho}} \partial_a \rho$ ,  $\alpha \equiv \frac{1}{3} \int d\tau \nabla_a u^a$  is the local expansion. See [54] for a recent review.

horizon, which is just the time of emission  $\eta_e$  for our purpose. Our next task is to set the initial conditions for  $\Phi$  and  $\Psi$  in the matter dominated era up to the third-order in the primordial curvature perturbation  $\zeta$ .

In matter-dominated era ( $p = 0$ ), (3.1) can be integrated to give

$$\zeta = -\Psi + \frac{1}{3} \ln \frac{\rho_m}{\bar{\rho}_m}. \quad (3.2)$$

On large scales, the matter density  $\rho_m$  can be related to metric perturbations through Einstein equation as (see Appendix (B11)-(B13)):

$$\frac{\rho_m}{\bar{\rho}_m} = e^{-2\Phi}, \quad (3.3)$$

which implies [15, 16]

$$\zeta = -\Psi - \frac{2}{3}\Phi. \quad (3.4)$$

(3.4) is a non-perturbatively relation among  $\zeta$  and  $\Phi$ ,  $\Psi$  on large scales during matter era.

At linear order, for fluid without anisotropic stress,  $\Phi_{(1)} = \Psi_{(1)}$ , which gives the well-known relation  $\zeta = -\frac{5}{3}\Phi_{(1)}$ . However,  $\Phi \neq \Psi$  at non-linear orders even for perfect fluid [39, 40] (see also [41–44] for the discussion of evolution of higher-order cosmological perturbations). From the traceless part of  $(i-j)$ -component of Einstein equation and using  $(0-0)$  and  $(0-i)$  components to express  $\rho$  and  $u_i$  in terms of metric perturbations (see Appendix (B11)-(B12)), we are able to write a non-perturbative constraint between  $\Phi$  and  $\Psi$  on large scales during matter-dominated era [15, 16],

$$\begin{aligned} \partial^4 (\Psi - \Phi) = & \frac{7}{2} (\partial^2 \Phi)^2 - \frac{3}{2} (\partial^2 \Psi)^2 + \frac{7}{6} (\partial_i \partial_j \Phi)^2 - \frac{1}{2} (\partial_i \partial_j \Psi)^2 + \frac{14}{3} \partial_i \Phi \partial_i \partial^2 \Phi - 2 \partial_i \Psi \partial_i \partial^2 \Psi \\ & + \partial_i \partial_j \Phi \partial_i \partial_j \Psi + 3 \partial^2 \Phi \partial^2 \Psi + 2 \partial_i \Phi \partial_i \partial^2 \Psi + 2 \partial_i \Psi \partial_i \partial^2 \Phi. \end{aligned} \quad (3.5)$$

In deriving (3.5) we have neglected higher-order spatial derivative terms since we are focusing on large scales. From (3.5)  $\Psi$  can be solved up to third-order in  $\Phi$  as

$$\Psi = \Phi + \partial^{-4} \left[ 5 (\partial^2 \Phi)^2 + \frac{5}{3} (\partial_i \partial_j \Phi)^2 + \frac{20}{3} \partial_i \Phi \partial_i \partial^2 \Phi \right]. \quad (3.6)$$

This is the generalization of the linear-order relation  $\Psi = \Phi$  up to the third-order in  $\Phi$ . It is interesting to note the third-order part of (3.6) exactly vanishes.

Combining (3.4) and (3.6), it is now straightforward to solve  $\Phi = \Phi[\zeta] = \Phi_{(1)} + \Phi_{(2)} + \Phi_{(3)} + \dots$  perturbatively to give

$$\Phi_{(1)} = -\frac{3}{5}\zeta, \quad (3.7)$$

$$\Phi_{(2)} = -\frac{9}{25} \partial^{-4} \left[ 3 (\partial^2 \zeta)^2 + (\partial_i \partial_j \zeta)^2 + 4 \partial_i \zeta \partial_i \partial^2 \zeta \right], \quad (3.8)$$

and

$$\begin{aligned} \Phi_{(3)} = & -\frac{54}{125} \partial^{-4} \left[ (3 \partial^2 \zeta \partial^{-2} + \partial_i \partial_j \zeta \partial_i \partial_j \partial^{-4} + 2 \partial_i \zeta \partial_i \partial^{-2} + 2 \partial_i \partial^2 \zeta \partial_i \partial^{-4}) \right. \\ & \left. \times \left( 3 (\partial^2 \zeta)^2 + (\partial_i \partial_j \zeta)^2 + 4 \partial_i \zeta \partial_i \partial^2 \zeta \right) \right]. \end{aligned} \quad (3.9)$$

(3.7)-(3.9) give the large-scale initial condition for  $\Phi$  during the matter era, in terms of the conserved primordial curvature perturbation. In the above  $\partial^{-2}$  etc. can be understood in momentum space. From (3.7)-(3.9) and (3.6) the corresponding initial condition  $\Psi = \Psi[\zeta]$  up to the third-order in  $\zeta$  can also be easily get.

## B. Non-linear temperature anisotropy

In the last part of this note, we will relate the observed temperature anisotropy  $\frac{\Delta T}{T}$  to the primordial curvature perturbation  $\zeta$ , on large scales. To this end, we also need the intrinsic temperature at the emission surface in terms of metric perturbations. A fully treatment involves dynamics during recombination [31–38]. Here in this note, we take the large-scale non-perturbative expression found in [16], where in matter dominated era with adiabatic assumption:  $T_e = \bar{T}_e e^{-\frac{2}{3}\Phi}$ . Thus the large-scale temperature anisotropy up to the third-order in  $\Phi$  is given by  $\frac{\Delta T}{T} = \left(\frac{\Delta T}{T}\right)_{(1)} + \left(\frac{\Delta T}{T}\right)_{(2)} + \left(\frac{\Delta T}{T}\right)_{(3)} + \dots$  with

$$\begin{aligned} \left(\frac{\Delta T}{T}\right)_{(1)} &= \frac{\Phi}{3} - I_1, \\ \left(\frac{\Delta T}{T}\right)_{(2)} &= \frac{\Phi^2}{18} + \frac{1}{3} \partial_i \Phi \left( n^i x_{(1)}^0 + x_{(1)}^i \right) - \frac{\Phi I_1}{3} - I_2 + x_{(1)}^0 A', \end{aligned} \quad (3.10)$$



and

$$\begin{aligned} \left(\frac{\Delta T}{T}\right)_{(3)} &= \frac{\Phi^3}{162} - \frac{\Phi^2 I_1}{18} + x_{(2)}^0 A' + \frac{1}{3} \Phi \left( x_{(1)}^0 A' - I_2 \right) - I_3 + \frac{1}{2} x_{(1)}^0 \left[ \partial_i A' \left( n^i x_{(1)}^0 + 2x_{(1)}^i \right) - 2I_1 A' + x_{(1)}^0 A'' \right] \\ &+ \partial_i \Phi \left[ \frac{1}{3} \left( x_{(2)}^i + x_{(1)}^0 I_1^i \right) + \frac{1}{9} n^i \left( 3x_{(2)}^0 + x_{(1)}^0 \left( \Phi + 6A - 18x_{(1)}^0 - 6I_1 \right) \right) + x_{(1)}^i \left( \Phi - 3I_1 \right) \right] \\ &+ \frac{1}{6} \partial_i \partial_j \Phi \left( x_{(1)}^i + n^i x_{(1)}^0 \right) \left( x_{(1)}^j + n^j x_{(1)}^0 \right) + 2n^i \partial_i \Phi' \left( x_{(1)}^0 \right)^2. \end{aligned} \quad (3.11)$$

In the following, in order to further relate  $\frac{\Delta T}{T}$  to  $\zeta$ , we take the SW contribution [16] where we neglect ISW and lensing contributions:  $\frac{\Delta T}{T} = \frac{\Phi}{3} + \frac{\Phi^2}{18} + \frac{\Phi^3}{162} + \dots \simeq e^{\frac{\Phi}{3}} - 1$ . Using (3.7)-(3.9), in momentum space, the non-linear mapping from  $\zeta$  to  $\frac{\Delta T}{T}$  up to the third-order in  $\zeta$  can be easily get. At linear order we find the familiar relation  $\left(\frac{\Delta T}{T}\right)_{(1)} = -\frac{1}{5}\zeta$ , at the second-order and third-order we find

$$\left(\frac{\Delta T}{T}\right)_{(2)}(\mathbf{k}) = \frac{1}{2} \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^3} \delta^3(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2) \beta(k; p_1, p_2) \zeta_{\mathbf{p}_1} \zeta_{\mathbf{p}_2}, \quad (3.12)$$

$$\left(\frac{\Delta T}{T}\right)_{(3)}(\mathbf{k}) = \frac{1}{3!} \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{(2\pi)^6} \delta^3(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \gamma(\mathbf{k}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \zeta_{\mathbf{p}_1} \zeta_{\mathbf{p}_2} \zeta_{\mathbf{p}_3}, \quad (3.13)$$

where the kernel

$$\beta(k; p_1, p_2) = -\frac{1}{50} + \frac{9(p_1^2 - p_2^2)^2}{50k^4} - \frac{3(p_1^2 + p_2^2)}{25k^2}, \quad (3.14)$$

$$\gamma(\mathbf{k}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = -\frac{1}{125} + [(1 - g(\mathbf{p}_1, \mathbf{p}_{23})) g(\mathbf{p}_2, \mathbf{p}_3) + 2 \text{cyclic}], \quad (3.15)$$

with  $p_{ij} = p_i + p_j$  and

$$g(\mathbf{p}, \mathbf{q}) = \frac{3}{250} \left[ 1 + 2 \frac{p^2 + q^2}{(\mathbf{p} + \mathbf{q})^2} - 3 \frac{(p^2 - q^2)^2}{(\mathbf{p} + \mathbf{q})^4} \right]. \quad (3.16)$$

It is interesting to note that  $\beta \rightarrow 1/25$  and  $g \rightarrow 0$  when  $\mathbf{p} \rightarrow 0$  or  $\mathbf{q} \rightarrow 0$ , which implies that the non-linear mapping (3.12)-(3.13) would not contribute significantly in the so-called “squeezed” momenta configurations of non-Gaussianity [13, 15, 16].

Using (A5) and (A6), the bispectrum and trispectrum for  $\frac{\Delta T}{T}$  can be read as

$$B(k_1, k_2, k_3) = \alpha^3 B_\zeta(k_1, k_2, k_3) + (\alpha^2 P_\zeta(k_1) P_\zeta(k_2) \beta(k_3; k_1, k_2) + 2 \text{cyclic}) \quad (3.17)$$

and

$$\begin{aligned} T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) &= \alpha^4 T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \\ &+ \alpha^3 B_\zeta(k_1, k_2, k_{12}) P_\zeta(k_3) \beta(k_4; k_{12}, k_3) + 11 \text{perms} \\ &+ \alpha^3 P_\zeta(k_1) P_\zeta(k_2) P_\zeta(k_3) \gamma(\mathbf{k}_4; -\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3) + 3 \text{perms} \\ &+ \alpha^2 P_\zeta(k_1) P_\zeta(k_2) P_\zeta(k_{13}) \beta(k_3; k_1, k_{13}) \beta(k_4; k_2, k_{13}) + 11 \text{perms}. \end{aligned} \quad (3.18)$$

Here  $B_\zeta$  and  $T_\zeta$  are primordial bispectrum and trispectrum for the curvature perturbation  $\zeta$  respectively.

In [6, 56], the non-Gaussianities are conventionally characterized by non-linear relation of the Bardeen potential<sup>6</sup>

$$\Phi = \Phi_L + f_{\text{NL}} * \Phi_L^2 + g_{\text{NL}} * \Phi_L^3, \quad (3.19)$$

where  $\Phi_L$  is the Gaussian part of  $\Phi$  and  $f_{\text{NL}}$  and  $g_{\text{NL}}$  are the so-called non-linear parameters, “\*” denotes possible integration in momentum space<sup>7</sup>. To make contact with previous analysis and conventions in the literature, in the following we use the linear-order relations during matter-dominated era:  $\frac{\Delta T}{T} \equiv -\frac{1}{3}\Phi$  and  $\frac{1}{3}\Phi_L = \frac{1}{5}\zeta_L$ , and make the ansatz for primordial non-Gaussianity:  $\zeta = \zeta_L + \frac{3}{5}f_{\text{NL}}^\zeta * \zeta_L^2 + \frac{9}{25}g_{\text{NL}}^\zeta * \zeta_L^3$ . After some manipulations, the non-linear parameters defined in (3.19) can be calculated as

$$f_{\text{NL}}(k; p_1, p_2) = f_{\text{NL}}^\zeta(k; p_1, p_2) - \frac{25}{3} \beta(k; p_1, p_2), \quad (3.20)$$

<sup>6</sup> Here the Bardeen potential  $\Phi$  should not be confused with the metric perturbation  $\Phi$  in (2.2). Actually at linear order,  $\Phi_L = -\Phi_{(1)}$ .

<sup>7</sup> For example,  $f_{\text{NL}} * \Phi_L^2 \equiv \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^3} \delta^3(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2) f_{\text{NL}}(k; p_1, p_2) \Phi_L(\mathbf{p}_1) \Phi_L(\mathbf{p}_2)$ .

and

$$g_{\text{NL}}(\mathbf{k}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = g_{\text{NL}}^\zeta(\mathbf{k}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) - \frac{50}{9} \left[ \beta(k; p_1, |\mathbf{k} - \mathbf{p}_1|) f_{\text{NL}}^\zeta(|\mathbf{k} - \mathbf{p}_1|, p_2, p_3) + 2 \text{cyclic} \right] - \frac{125}{9} \gamma(\mathbf{k}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3). \quad (3.21)$$

where  $f_{\text{NL}}^\zeta$  and  $g_{\text{NL}}^\zeta$  are non-linear parameters for primordial curvature perturbation  $\zeta$ , the functions  $\beta$  and  $\gamma$  are given in (3.14) and (3.15). In (3.20)-(3.21), different contributions to the finally observed non-Gaussianity from primordial epoch and from non-linearity between  $\zeta$  and  $\frac{\Delta T}{T}$  are transparent.

#### IV. CONCLUSION

In this note, following the approach developed in [8–10], the general expression for the observed CMB anisotropy up to the third-order is calculated [15]. In the Sachs-Wolfe limit, we derive the non-linear relation between the observed anisotropy and the conserved primordial curvature perturbation  $\zeta$ , up to the third-order in  $\zeta$ . (3.12)-(3.13) can be viewed as non-linear generalization of familiar linear relation  $\Delta T/T = -\zeta/5$ . Our formalism is valid for large-scale anisotropies, which re-enter the horizon after decoupling. The results (3.17)-(3.18) clearly show the different contributions to the observed non-Gaussianity from primordial non-Gaussianities in  $\zeta$  and non-linear mapping from  $\zeta$  to  $\frac{\Delta T}{T}$  due to gravitational perturbations. We also derive the non-linear parameters  $f_{\text{NL}}$  and  $g_{\text{NL}}$  (eq.(3.20)-(3.21)), which enter the theoretical predictions for the angular bispectrum and trispectrum of CMB respectively.

We do not expect the non-linear mapping in the SW limit (3.12)-(3.13) would give a major contribution to the observed non-Gaussianities, especially comparing scenarios where large “primordial” non-Gaussianities can be generated (see [58] for a recent review and references therein). However, the other secondary anisotropies, especially the correlation between lensing and ISW effect which we do not discuss in this note, is expected to give contribution to the final non-Gaussianity [17, 21–26]. Another interesting issue is that, if an enhancement of non-linearity between  $\frac{\Delta T}{T}$  and  $\zeta$  is possible, like the enhancement of primordial non-Gaussianity by small  $c_s$  etc. in some inflationary scenarios. We wish to come back to these subjects in future investigations.

#### Acknowledgments

I appreciate Yi-Fu Cai, Tao-Tao Qiu and Yi Wang for useful discussion and comments. I am grateful to Prof. Miao Li for consistent encouragement and support. This work was supported by the NSFC grant No.10535060/A050207, a NSFC group grant No.10821504 and Ministry of Science and Technology 973 program under grant No.2007CB815401.

#### Appendix A: Non-Gaussianities from non-linear mapping

Non-Gaussian variables can be get from non-linear mapping from Gaussian/non-Gaussian variables. The  $\delta N$ -formalism is a non-linear mapping from inflaton fluctuation  $\delta\phi$  to the curvature perturbation  $\zeta$  on super-Hubble scales. In general, the non-linear mapping in real space from a single variable  $Q$  to  $\zeta$  comes from three types: local products (e.g.  $\delta N$ -formalism), products of local derivative terms, products of non-local terms. In all cases, the mapping can be written in fourier space as:

$$\zeta_{\mathbf{k}} = \alpha Q_{\mathbf{k}} + \frac{1}{2!} \int \widetilde{dp_1} \widetilde{dp_2} \tilde{\beta}_{\mathbf{k}; p_1, p_2} Q_{\mathbf{p}_1} Q_{\mathbf{p}_2} + \frac{1}{3!} \int \widetilde{dp_1} \widetilde{dp_2} \widetilde{dp_3} \tilde{\gamma}_{\mathbf{k}; p_1, p_2, p_3} Q_{\mathbf{p}_1} Q_{\mathbf{p}_2} Q_{\mathbf{p}_3} + \dots, \quad (A1)$$

with  $\widetilde{dp} \equiv \frac{d^3 p}{(2\pi)^3}$ ,  $\alpha$  is a  $\mathbf{k}$ -independent number and

$$\tilde{\beta}_{\mathbf{k}; p_1, p_2} \equiv (2\pi)^3 \delta(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2) \beta(k; p_1, p_2), \quad (A2)$$

$$\tilde{\gamma}_{\mathbf{k}; p_1, p_2, p_3} \equiv (2\pi)^3 \delta(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \gamma(\mathbf{k}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3), \quad (A3)$$

where  $\beta(k; p_1, p_2)$  and  $\gamma(\mathbf{k}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$  are normal functions. It is useful to note that  $\beta$  and  $\gamma$  are symmetric with respect to all  $\mathbf{p}_i$ 's.

It is more convenient to calculate the correlation functions of  $\hat{\zeta} \equiv \zeta - \langle \zeta \rangle$  since  $\langle \hat{\zeta} \rangle = 0$ . Straightforward calculation gives (subscript “c” denotes connected contribution):

$$\begin{aligned} \langle \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \hat{\zeta}_{\mathbf{k}_3} \rangle &= (2\pi)^3 \delta^3 \left( \sum_{i=1}^3 \mathbf{k}_i \right) B_\zeta(k_1, k_2, k_3), \\ \langle \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \hat{\zeta}_{\mathbf{k}_3} \hat{\zeta}_{\mathbf{k}_4} \rangle_c &= (2\pi)^3 \delta \left( \sum_{i=1}^4 \mathbf{k}_i \right) T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4), \end{aligned} \quad (A4)$$

with the leading contributions:

$$B_\zeta(k_1, k_2, k_3) = \alpha^3 B_Q(k_1, k_2, k_3) + (\alpha^2 P_Q(k_1) P_Q(k_2) \beta(k_3; k_1, k_2) + \text{cyclic}), \quad (A5)$$



and

$$\begin{aligned}
T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = & \alpha^4 T_Q(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \\
& + \alpha^3 B_Q(k_1, k_2, k_{12}) P_Q(k_3) \beta(k_4; k_{12}, k_3) + 11 \text{perms} \\
& + \alpha^3 P_Q(k_1) P_Q(k_2) P_Q(k_3) \gamma(\mathbf{k}_4; -\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3) + 3 \text{perms} \\
& + \alpha^2 P_Q(k_1) P_Q(k_2) P_Q(k_{13}) \beta(k_3; k_1, k_{13}) \beta(k_4; k_2, k_{13}) + 11 \text{perms}
\end{aligned} \tag{A6}$$

with  $k_{ij} \equiv |\mathbf{k}_i + \mathbf{k}_j|$ . In the above  $P_Q$  is the power spectrum of  $Q$ ,  $B_Q$  and  $T_Q$  are “intrinsic” bispectrum and trispectrum for  $Q$  respectively which vanish if  $Q$  is purely Gaussian.

In [55], based on the  $\delta N$ -formalism, Feynman-type diagrams were introduced to represent various contributions to non-Gaussianity from the non-linear mapping from  $\delta\phi$  to  $\zeta$  on large scales. In general this can be generalized straightforwardly for the non-linear mapping (A1), as we show in fig.1 and fig.2.

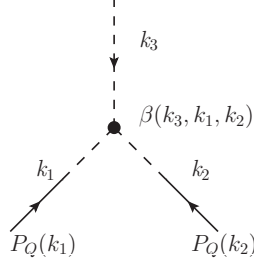


FIG. 1: Diagrammatic representation of  $P(k_1)P(k_2)\beta(k_3, k_1, k_2)$  in (A5).

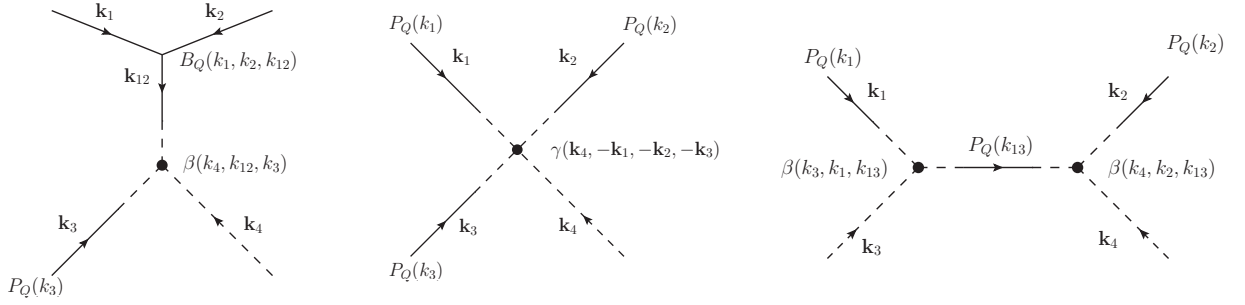


FIG. 2: Diagrammatic representation of the second, third and fourth line in (A6).

## Appendix B: Perturbed metric and related quantities

For the perturbed conformal metric  $g_{\mu\nu}$  defined in (2.2), the components of Christoffel symbol are

$$\Gamma_{00}^0 = \frac{e^{2\Phi}}{N^2} \left[ \Phi' + e^{2(\Psi-\Phi)} e^{-\gamma_{ij}} \sigma_j (\sigma'_i + e^{2\Phi} \partial_i \Phi) \right], \tag{B1}$$

$$\Gamma_{0i}^0 = \Gamma_{i0}^0 = \frac{e^{2\Phi}}{N^2} \left\{ \partial_i \Phi + \frac{1}{2} e^{2(\Psi-\Phi)} e^{-\gamma_{jm}} \sigma_m \left[ e^{-2\Psi} (-2\Psi' e^{\gamma_{ij}} + (e^{\gamma_{ij}})') + \partial_i \sigma_j - \partial_j \sigma_i \right] \right\}, \tag{B2}$$

$$\begin{aligned}
\Gamma_{ij}^0 = & \frac{e^{-2\Phi}}{2N^2} \left\{ -\partial_i \sigma_j - \partial_j \sigma_i - 2\Psi' e^{-2\Psi} e^{\gamma_{ij}} + e^{-2\Psi} (e^{\gamma_{ij}})' \right. \\
& \left. - e^{-\gamma_{kl}} \sigma_l [(2\partial_i \Psi e^{\gamma_{jk}} - \partial_i e^{\gamma_{jk}}) + (2\partial_j \Psi e^{\gamma_{ki}} - \partial_j e^{\gamma_{ki}}) - (2\partial_k \Psi e^{\gamma_{ij}} - \partial_k e^{\gamma_{ij}})] \right\}.
\end{aligned} \tag{B3}$$

$$\Gamma_{00}^i = -\frac{e^{2(\Phi+\Psi)}}{N^2} \Phi' e^{-\gamma_{ij}} \sigma_j + e^{2\Psi} e^{-\gamma_{ik}} \left( \delta_{kj} - \frac{e^{2\Psi} \sigma_k e^{-\gamma_{jn}} \sigma_n}{N^2} \right) (\sigma'_j + e^{2\Phi} \partial_j \Phi), \tag{B4}$$

$$\begin{aligned}
\Gamma_{0j}^i & \equiv \Gamma_{j0}^i \\
= & -\frac{e^{2(\Phi+\Psi)}}{N^2} e^{-\gamma_{ik}} \sigma_k \partial_j \Phi + \frac{1}{2} \left( e^{-\gamma_{ik}} - \frac{e^{2\Psi}}{N^2} (e^{-\gamma_{im}} \sigma_m) (e^{-\gamma_{kn}} \sigma_n) \right) [(-2\Psi' e^{\gamma_{jk}} + (e^{\gamma_{jk}})') + e^{2\Psi} (\partial_j \sigma_k - \partial_k \sigma_j)],
\end{aligned} \tag{B5}$$

$$\Gamma_{jk}^i = \frac{1}{2} \frac{e^{-\gamma_{il}} \sigma_l}{N^2} [e^{2\Psi} (\partial_j \sigma_k + \partial_k \sigma_j) + 2\Psi' e^{\gamma_{jk}} - (e^{\gamma_{jk}})'] - \frac{1}{2} \left( e^{-\gamma_{il}} - \frac{e^{2\Psi}}{N^2} (e^{-\gamma_{im}} \sigma_m) (e^{-\gamma_{ln}} \sigma_n) \right) [(2\partial_j \Psi e^{\gamma_{kl}} - \partial_j e^{\gamma_{kl}}) + (2\partial_k \Psi e^{\gamma_{lj}} - \partial_k e^{\gamma_{lj}}) - (2\partial_l \Psi e^{\gamma_{jk}} - \partial_l e^{\gamma_{jk}})]. \quad (\text{B6})$$

In the above  $N^2 = e^{2\Phi} + e^{2\Psi} e^{-\gamma_{ij}} \sigma_i \sigma_j$ , where  $N$  is the corresponding lapse function in ADM formalism.

For metric  $ds^2 = a^2 (-e^{2\Phi} d\eta^2 + e^{-2\Psi} dx^i dx^i)$ , the Christoffel connection is significantly simplified, with components:

$$\begin{aligned} \Gamma_{00}^0 &= \Phi', & \Gamma_{0i}^0 &= \partial_i \Phi, & \Gamma_{ij}^0 &= -\Psi' e^{-2(\Phi+\Psi)} \delta_{ij}, & \Gamma_{00}^i &= e^{2(\Phi+\Psi)} \partial_i \Phi, \\ \Gamma_{0j}^i &= -\Psi' \delta_{ij}, & \Gamma_{jk}^i &= -(\partial_j \Psi \delta_{ki} + \partial_k \Psi \delta_{ij} - \partial_i \Psi \delta_{jk}). \end{aligned} \quad (\text{B7})$$

The corresponding components of Einstein tensor are

$$G_{00} = 3(\mathcal{H} - \Psi')^2 - e^{2(\Phi+\Psi)} \left( (\partial_i \Psi)^2 - 2\partial^2 \Psi \right), \quad (\text{B8})$$

$$G_{0i} = 2[\partial_i \Psi' + \partial_i \Phi (\mathcal{H} - \Psi')], \quad (\text{B9})$$

$$\begin{aligned} G_{ij} &= \left\{ e^{-2(\Phi+\Psi)} [-2\mathcal{H}' - (\mathcal{H} - 2\Phi' - 3\Psi') (\mathcal{H} - \Psi') + 2\Psi''] + (\partial_i \Phi)^2 + \partial^2 (\Phi - \Psi) \right\} \delta_{ij} \\ &\quad + \partial_i \partial_j (\Psi - \Phi) - \partial_i \Phi \partial_j \Phi + \partial_i \Psi \partial_j \Psi - (\partial_i \Phi \partial_j \Psi + \partial_i \Psi \partial_j \Phi). \end{aligned} \quad (\text{B10})$$

The above expressions are exact, which can be easily expanded to the desired orders.

The (00)-component of Einstein equation  $G_{\mu\nu} = T_{\mu\nu}$  gives,

$$\frac{\rho_m}{\bar{\rho}_m} = \frac{3(\mathcal{H} - \Psi')^2 - e^{2(\Phi+\Psi)} \left( (\partial_i \Psi)^2 - 2\partial^2 \Psi \right)}{1 + a^{-2} e^{2\Psi} u_i u_i} \frac{e^{-2\Phi}}{3\mathcal{H}^2}, \quad (\text{B11})$$

where we used  $\bar{\rho}_m = \frac{3\mathcal{H}^2}{a^2}$  on the background level. From the (0i)-component of Einstein equation we can solve

$$1 + a^{-2} e^{2\Psi} u_i u_i = \frac{\left[ 3\mathcal{H}^2 - e^{2(\Phi+\Psi)} \left( (\partial_i \Psi)^2 - 2\partial^2 \Psi \right) \right]^2}{\left[ 3\mathcal{H}^2 - e^{2(\Phi+\Psi)} \left( (\partial_i \Psi)^2 - 2\partial^2 \Psi \right) \right]^2 - 4\mathcal{H}^2 e^{2(\Phi+\Psi)} (\partial_i \Phi)^2}. \quad (\text{B12})$$

where we have set  $\Psi' = 0$ . Finally, (B11) and (B12) imply

$$\frac{\rho_m}{\bar{\rho}_m} = e^{-2\Phi} \left[ 1 - \frac{e^{2(\Phi+\Psi)}}{3\mathcal{H}^2} \left( (\partial_i \Psi)^2 - 2\partial^2 \Psi \right) - \frac{4e^{2(\Phi+\Psi)}}{9\mathcal{H}^2} \frac{(\partial_i \Phi)^2}{1 - \frac{e^{2(\Phi+\Psi)}}{3\mathcal{H}^2} \left( (\partial_i \Psi)^2 - 2\partial^2 \Psi \right)} \right], \quad (\text{B13})$$

which gives, on large scales,  $\rho_m / \bar{\rho}_m \simeq e^{-2\Phi}$ .

- 
- [1] E. Komatsu *et al.*, arXiv:1001.4538 [astro-ph.CO].
  - [2] E. Komatsu, arXiv:1003.6097 [astro-ph.CO].
  - [3] D. H. Lyth and A. Riotto, Phys. Rept. **314** (1999) 1 [arXiv:hep-ph/9807278].
  - [4] N. Bartolo, E. Komatsu, S. Matarrese and A. Riotto, Phys. Rept. **402**, 103 (2004) [arXiv:astro-ph/0406398].
  - [5] N. Bartolo, S. Matarrese and A. Riotto, arXiv:1001.3957 [astro-ph.CO].
  - [6] E. Komatsu and D. N. Spergel, Phys. Rev. D **63** (2001) 063002 [arXiv:astro-ph/0005036].
  - [7] T. Pyne and M. Birkinshaw, arXiv:astro-ph/9303020.
  - [8] T. Pyne and S. M. Carroll, Phys. Rev. D **53**, 2920 (1996) [arXiv:astro-ph/9510041].
  - [9] S. Mollerach and S. Matarrese, Phys. Rev. D **56**, 4494 (1997) [arXiv:astro-ph/9702234].
  - [10] S. Matarrese, S. Mollerach and M. Bruni, Phys. Rev. D **58**, 043504 (1998) [arXiv:astro-ph/9707278].
  - [11] R. Maartens, T. Gebbie and G. F. R. Ellis, Phys. Rev. D **59**, 083506 (1999) [arXiv:astro-ph/9808163].
  - [12] P. K. S. Dunsby, Class. Quant. Grav. **14**, 3391 (1997) [arXiv:gr-qc/9707022].
  - [13] L. Boubekeur, P. Creminelli, G. D'Amico, J. Norena and F. Vernizzi, JCAP **0908**, 029 (2009) [arXiv:0906.0980 [astro-ph.CO]].
  - [14] N. Bartolo, S. Matarrese and A. Riotto, Phys. Rev. Lett. **93**, 231301 (2004) [arXiv:astro-ph/0407505].
  - [15] G. D'Amico, N. Bartolo, S. Matarrese and A. Riotto, JCAP **0801**, 005 (2008) [arXiv:0707.2894 [astro-ph]].
  - [16] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0508** (2005) 010 [arXiv:astro-ph/0506410].
  - [17] A. Cooray, Phys. Rev. D **65** (2002) 083518 [arXiv:astro-ph/0109162].
  - [18] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0605**, 010 (2006) [arXiv:astro-ph/0512481].
  - [19] K. Tomita, Phys. Rev. D **71**, 083504 (2005) [arXiv:astro-ph/0501663].

- [20] K. Tomita and K. T. Inoue, Phys. Rev. D **77** (2008) 103522 [arXiv:0712.1291 [astro-ph]].
- [21] U. Seljak and M. Zaldarriaga, Phys. Rev. D **60** (1999) 043504 [arXiv:astro-ph/9811123].
- [22] D. M. Goldberg and D. N. Spergel, Phys. Rev. D **59**, 103002 (1999) [arXiv:astro-ph/9811251].
- [23] P. Serra and A. Cooray, Phys. Rev. D **77**, 107305 (2008) [arXiv:0801.3276 [astro-ph]].
- [24] D. Hanson, K. M. Smith, A. Challinor and M. Liguori, Phys. Rev. D **80**, 083004 (2009) [arXiv:0905.4732 [astro-ph.CO]].
- [25] A. Mangilli and L. Verde, Phys. Rev. D **80**, 123007 (2009) [arXiv:0906.2317 [astro-ph.CO]].
- [26] P. Creminelli and M. Zaldarriaga, Phys. Rev. D **70** (2004) 083532 [arXiv:astro-ph/0405428].
- [27] R. Khatri and B. D. Wandelt, Phys. Rev. D **79**, 023501 (2009) [arXiv:0810.4370 [astro-ph]].
- [28] L. Senatore, S. Tashev and M. Zaldarriaga, JCAP **0908**, 031 (2009) [arXiv:0812.3652 [astro-ph]].
- [29] L. Senatore, S. Tashev and M. Zaldarriaga, JCAP **0909**, 038 (2009) [arXiv:0812.3658 [astro-ph]].
- [30] N. Bartolo and A. Riotto, JCAP **0903**, 017 (2009) [arXiv:0811.4584 [astro-ph]].
- [31] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0606** (2006) 024 [arXiv:astro-ph/0604416].
- [32] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0701** (2007) 019 [arXiv:astro-ph/0610110].
- [33] D. Nitta, E. Komatsu, N. Bartolo, S. Matarrese and A. Riotto, JCAP **0905**, 014 (2009) [arXiv:0903.0894 [astro-ph.CO]].
- [34] C. Pitrou, J. P. Uzan and F. Bernardeau, Phys. Rev. D **78**, 063526 (2008) [arXiv:0807.0341 [astro-ph]].
- [35] C. Pitrou, Class. Quant. Grav. **26** (2009) 065006 [arXiv:0809.3036 [gr-qc]].
- [36] C. Pitrou, Gen. Rel. Grav. **41** (2009) 2587 [arXiv:0809.3245 [astro-ph]].
- [37] C. Pitrou, J. P. Uzan and F. Bernardeau, arXiv:1003.0481 [astro-ph.CO].
- [38] M. Beneke and C. Fidler, arXiv:1003.1834 [astro-ph.CO].
- [39] V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, Nucl. Phys. B **667** (2003) 119 [arXiv:astro-ph/0209156].
- [40] N. Bartolo, S. Matarrese and A. Riotto, JHEP **0404** (2004) 006 [arXiv:astro-ph/0308088].
- [41] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0401** (2004) 003 [arXiv:astro-ph/0309692].
- [42] N. Bartolo, S. Matarrese and A. Riotto, Phys. Rev. D **69** (2004) 043503 [arXiv:hep-ph/0309033].
- [43] L. Boubekur, P. Creminelli, J. Norena and F. Vernizzi, JCAP **0808** (2008) 028 [arXiv:0806.1016 [astro-ph]].
- [44] A. J. Christopherson and K. A. Malik, JCAP **0911** (2009) 012 [arXiv:0909.0942 [astro-ph.CO]].
- [45] D. S. Salopek and J. R. Bond, Phys. Rev. D **42** (1990) 3936.
- [46] E. W. Kolb, S. Matarrese, A. Notari and A. Riotto, Mod. Phys. Lett. A **20** (2005) 2705 [arXiv:astro-ph/0410541].
- [47] D. H. Lyth, K. A. Malik and M. Sasaki, JCAP **0505** (2005) 004 [arXiv:astro-ph/0411220].
- [48] G. I. Rigopoulos and E. P. S. Shellard, Phys. Rev. D **68** (2003) 123518 [arXiv:astro-ph/0306620].
- [49] K. A. Malik and D. Wands, Class. Quant. Grav. **21** (2004) L65 [arXiv:astro-ph/0307055].
- [50] D. H. Lyth and D. Wands, Phys. Rev. D **68** (2003) 103515 [arXiv:astro-ph/0306498].
- [51] D. H. Lyth and Y. Rodriguez, Phys. Rev. D **71** (2005) 123508 [arXiv:astro-ph/0502578].
- [52] D. Langlois and F. Vernizzi, Phys. Rev. Lett. **95** (2005) 091303 [arXiv:astro-ph/0503416].
- [53] D. Langlois and F. Vernizzi, Phys. Rev. D **72** (2005) 103501 [arXiv:astro-ph/0509078].
- [54] D. Langlois and F. Vernizzi, arXiv:1003.3270 [astro-ph.CO].
- [55] C. T. Byrnes, K. Koyama, M. Sasaki and D. Wands, JCAP **0711** (2007) 027 [arXiv:0705.4096 [hep-th]].
- [56] E. Komatsu *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148** (2003) 119 [arXiv:astro-ph/0302223].
- [57] J. M. Maldacena, JHEP **0305**, 013 (2003) [arXiv:astro-ph/0210603].
- [58] K. Koyama, arXiv:1002.0600 [hep-th].
- [59] X. Gao, work in progress.